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Abstract. Market clearing in auction-based procurement processes face a fundamental problem: the market-maker does not have direct access to information about the bidders' private preferences, as well as their specific constraints. This issue has motivated the development of iterative auction mechanisms that induce the bidders into progressively revealing their private preferences. In this paper, we consider a model that abstracts in a fairly general way an exchange of interdependent goods and we propose an iterative auction mechanism based on the well-known Dantzig-Wolfe decomposition principle, where the bidders reveal parts of their preferences through straightforward, utility-maximizing bids. Our proposed auction mechanism is illustrated on a simulated wood chip market and numerical results obtained are contrasted with those of auctions based on Lagrangian relaxation.

Keywords: Procurement, mechanism design, auction mechanisms, mathematical programming, decomposition methods.

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1 INTRODUCTION AND BACKGROUND

The rise of electronic commerce and Business-to-Business procurement platforms was accompanied by renewed interest in the design of market mechanisms. Used from times immemorial as support for selling goods and services, auctions are perhaps the most popular market mechanism known to the human kind. Traditionally, microeconomic theory sets as chief target the design of direct revelation market mechanisms (MasColell, Whinston, and Green [21]) where it is assumed bidders will report truthfully their preferences to a market-maker in charge of clearing the market while pursuing an economic objective (e.g., maximize the overall efficiency). A significant portion of auction theory is dedicated to the design of auctions as practical implementations of direct revelation market mechanisms. The latter fall into two categories: (i) one-shot, sealed-bid auctions or (ii) iterative, multi-round auctions. Thanks to the latter's superior characteristics in terms of price elicitation, decentralization of information, and respect of private valuation, iterative auction design has attracted the most attention from recent research efforts.

Most of the recent literature on the design of iterative auction mechanisms has been directed toward an unilateral, indivisible goods, combinatorial auction setting. That is one seller selling several heterogeneous goods to many buyers, with the possibility for the latter to bid on bundle of goods. We shall present a brief overview of this branch of literature in Section 2, while directing the interested reader to more extensive surveys (e.g., De Vries and Vohra [16]; Mishra [22]). An interesting alternative is offered by mathematical programming decomposition approaches, which come with a legacy of addressing large-scale structured optimization problems. Yet, their potential for decentralized decision making, to the best of our knowledge, has not been fully exploited in the design of iterative auction mechanisms. This paper aspires to contribute to filling this gap by focusing on one of the most well-known and used of these methods, Dantzig-Wolfe (DW) decomposition, presented and interpreted as an iterative auction mechanism. The paper's starting point is a general combinatorial exchange economy in which the participants (sellers and buyers) trade heterogeneous but interdependent divisible commodities. The participants are assumed to be self-interested in the sense that they seek to maximize their own economic surplus by submitting optimal bids in reaction to prices announced by the market maker. The paper shows that the application of the DW decomposition to the centralized efficiency-maximizing allocation model leads to an iterative auction mechanism that has the ability to achieve social efficiency without requiring complete information revelation from the participants. The numerical efficiency of the mechanism, measured in terms of number of bidding rounds and CPU time needed for convergence is evaluated on a simulated wood chip market. Numerical comparison with another family of auction mechanisms based on Lagrangian relaxation (Abrache et al. [1]) is also proposed. More specifically, The paper's contribution is twofold: first, it aims to establish that the DW decomposition principle can form the basis of an auction mechanism for both a generic multilateral market of divisible goods and a particular procurement

case inspired from real wood chip markets; second, it contrasts numerically the proposed DW auction with dual Lagrangian relaxation based auctions in terms of: a) number of bidding rounds and CPU time to convergence, b) the extent of which a bidding round is numerically taxing, and c) the potential of stopping the bidding process before the usual criteria are met. The findings could shed much needed light on practical implementation aspects of decomposition based auctions, which to the best of our knowledge, have not been studied in the past.

The paper is organized as follows. We present in Section 2 an overview of the state of the art in iterative auction mechanism design. Section 3 presents the centralized model of socially efficient allocation that will be the basis of DW decomposition. In Section 4, we present and analyze our DW auction. Finally, we devote Section 5 to the experimental study and numerical analysis of results.

2 PRIOR WORK

Thanks to its numerous desirable properties, which include chiefly incentive compatibility, the Vickrey-Clarke-Groves (VCG) mechanism (Vickrey [27], Clarke [11], Groves [18]) plays a pivotal role in economic theory. The work on the design of iterative VCG auction mechanisms that would circumvent the requirement of fully communicating preferences to the auctioneer has been at the heart of modern auction theory research. In this particular line, the Combinatorial Auction Problem (CAP) has attracted most interest. The CAP is the problem of determining welfare-maximizing allocations of several indivisible items to buyers that have preferences for bundles of items. The existence of equilibrium prices for the CAP has been investigated and linked to the integrality of the LP relaxation of the CAP in Bikhchandani and Mamer [4]. Bikhchandani and Ostroy ([5],[6]) propose extended formulations of the CAP that imply nonlinear - and possibly discriminatory - bundle equilibrium prices. These advances formed the theoretical background needed for primal-dual implementations of auction mechanisms. The iBundle family of auctions (Parkes [24]) belongs to that category, as well as the iBEA (iBundle Extand and Adjust) Auction (Mishra and Parkes [23], Chen and Takeuchi [8]), which extends iBundle by determining universal competitive equilibrium prices that can be used to compute VCG payments.

Another branch of economic research concerns mechanisms based on the so-called tâtonnement process, originally suggested by Walras [28], and aimed at the determination of Walrasian equilibria. Generally speaking, Walrasian price-tâtonnement proceeds by adjusting the price of the various goods one by one until balance of supply and demand of all the goods is realized. Convergence is guaranteed if additional conditions for equilibrium *stability* are satisfied. For instance, the *gross-substitutes* condition, which states that the demand for a given item does not decrease if prices of other items increase,

is sufficient for equilibrium stability (Arrow and Hahn [2]). Examples of implementations of price-tâtonnement include the WALRAS algorithm (Cheng and Wellman [9]). Walrasian price-tâtonnement has also inspired computational paradigms such as *market-oriented programming* (Wellman [30]), which model and implement resource allocation problems as distributed systems of autonomous agents reacting to a pricing system. Applications of this paradigm include the allocation of transportation services (Wellman [29]), quality-of-service allocation in multimedia applications (Yamaki, Wellman, and Ishida [32]), vehicle routing (Sandholm [25]), power load management (Ygge [33]), and decentralized scheduling (Wellman et al. [31]). Quantity-tâtonnement processes form the other important class of market mechanisms based on tâtonnement. In an iteration of a quantity-tâtonnement process, the market-maker determines provisional allocations of goods and participants submit their marginal utilities; the allocations are then adjusted by the market-maker such that participants with higher marginal utilities are awarded larger quantities, and those with lower marginal utilities receive lower quantities; the process continues until an equilibrium state is reached. Specific quantity-tâtonnement mechanisms include simple fixed and variable step size adjustment methods suggested by Kurose and Simha [19], and a more efficient Newton-Raphson search method proposed by Ygge and Akkermans [34].

Since its introduction in Dantzig and Wolfe seminal papers [14, 15], DW decomposition has steadily attracted attention among academic circles and practitioners alike, with countless contributions exploring its theoretical aspects as well as applications. A concise modern survey of the fundamentals of DW decomposition can be found in Chung ([10]). Detailed expositions along with other decomposition methods (Lagrangian relaxation, Benders decomposition, etc.) appear in many classic references (e.g., Lasdon [20]) and in recent textbooks (e.g., Conejo et al. [12]). Applications of DW to specific optimization problems also form a rich literature (see for instance Vaidyanathan and Ahuja [26] for an application to the multicommodity flow problem and a comparative study with Lagrangian relaxation). Although the economic interpretation of DW decomposition as a decentralized decision making tool is well documented since the early contributions (Dantzig [13], Baumol and Fabian [3]), this decentralization has not been exploited to its full potential in auction design.

3 A CENTRALIZED MARKET CLEARING MODEL

We consider a simplified economy with a set of divisible goods on sale and two categories of participants, sellers and buyers. This model is the same as the one we previously used as the basis of an iterative auction mechanism we presented in Abrache et al. ([1]). In the model, there are several heterogeneous goods on sale. Sellers produce goods according to their specific technology and production cost functions, while buyers consume goods either directly or as intermediary inputs to a transformation process.

Buyers have preferences for *bundles* of goods and may also face technological requirements that constrain their consumption. The following notation is introduced:

- $\mathcal{L}$ (resp. $\mathcal{S}$, $\mathcal{J}$): set of goods (resp. sellers, buyers).
- $q_{s,l}$ (resp. $q_{j,l}$): quantity of good $l, l \in \mathcal{L}$ produced by seller s, $s \in \mathcal{S}$ (resp. consumed by buyer j, $j \in \mathcal{J}$).
- $\mathcal{D}_s$ (resp. $\mathcal{D}_j$): production (resp. consumption) feasibility set of seller s (resp. buyer j), containing all admissible quantities $q_s = \{q_{s,l}\}_{l\in\mathcal{L}}$ (resp. $q_j = \{q_{j,l}\}_{l\in\mathcal{L}}$) that seller s (resp. buyer j) may produce (resp. consume). These sets are assumed to be convex and bounded.
- $C_s(.)$: production cost function of seller s, $s \in \mathcal{S}$; that is, $C_s(q_s)$ is the cost to seller s of producing q_s. This cost function is assumed to be continuous, convex, and monotone increasing.
- $V_j(.)$: valuation function of buyer j, $j \in \mathcal{J}$; similarly, $V_j(q_j)$ is buyer j's preference for consuming q_j. This valuation function is assumed to be continuous, concave, and monotone increasing.

In a centralized market mechanism targeting overall efficiency, sellers and buyers need to communicate to the market-maker their production and consumption feasibility sets and their cost and valuation functions, respectively. The mechanism's output is an allocation of goods and payments sellers (resp. buyers) need to make (resp. receive). The market-maker needs to determine a *socially-efficient* allocation, that is, a feasible allocation of goods that maximizes the overall welfare of all sellers and buyers. More precisely, a socially-efficient allocation is a solution of model (MC):

$$\max \quad \sum_{j\in\mathcal{J}} V_j(q_j) - \sum_{s\in\mathcal{S}} C_s(q_s) \tag{1}$$

$$s.t. \quad \sum_{j\in\mathcal{J}} q_{j,l} - \sum_{s\in\mathcal{S}} q_{s,l} = 0, \quad l \in \mathcal{L} \tag{2}$$

$$q_j \in \mathcal{D}_j, \quad j \in \mathcal{J}; q_s \in \mathcal{D}_s, \quad s \in \mathcal{S} \tag{3}$$

Model (MC) maximizes the market surplus, that is, the difference between the buyers' valuations and the sellers' production costs. Constraints (2) match the demand with the supply, while constraints (3) are buyer and seller quantity feasibility constraints.

With the classical assumptions on the buyers and sellers of having quasi-linear utility functions and being price-takers, the concept of Walrasian Equilibrium can be defined.

Definition 1 *The allocation $\tilde{q} = [\{\tilde{q}_j\}_{j\in\mathcal{J}}; \{\tilde{q}_s\}_{s\in\mathcal{S}}]$ and the price vector $p = \{p_l\}_{l\in\mathcal{L}}$ form a Walrasian Equilibrium if: (1) The allocation $\tilde{q}$ is feasible for Model (MC); (2) $V_j(\tilde{q}_j) - p.\tilde{q}_j = \max_{q_j\in\mathcal{D}_j}(V_j(q_j) - p.q_j)$, $\forall j \in \mathcal{J}$; and (3) $p.\tilde{q}_s - C_s(\tilde{q}_s) = \max_{q_s\in\mathcal{D}_s}(p.q_s - C_s(q_s))$, $\forall s \in \mathcal{S}$.*

Condition 1 above means that $\tilde{q}$ is an acceptable allocation for all sellers and buyers, that matches the total supply with the total demand. Conditions (2) and (3) point out the behavior of sellers and buyers as price-taking, utility-maximizing participants.

4 A DW DECOMPOSITION AUCTION SCHEME

In the following, we establish that DW decomposition can be interpreted as an iterative auction providing a decentralized way to determine optimal allocations for model (MC). Assume sets of feasible consumption levels $\{q_j^r\}_{r\in k_j}$, $j \in \mathcal{J}$, and production levels $\{q_s^r\}_{r\in k_s}$, $s \in \mathcal{S}$ are initially available to the market-maker. Consider the corresponding convex hulls: $\tilde{\mathcal{D}}_j^{k_j} = \{q_j = \sum_{r=1}^{k_j} q_j^r\alpha_j^r : \sum_{r=1}^{k_j}\alpha_j^r = 1; \alpha_j^r \geq 0, \forall r = 1,\ldots,k_j\}$, $j \in \mathcal{J}$, and $\tilde{\mathcal{D}}_s^{k_s} = \{q_s = \sum_{r=1}^{k_s} q_s^r\beta_s^r : \sum_{r=1}^{k_s}\beta_s^r = 1; \beta_s^r \geq 0, \forall r = 1,\ldots,k_s\}$, $s \in \mathcal{S}$.

Inner-linearization (Geoffrion [17]) of the production (consumption) feasibility sets and the cost (valuation) functions suggests to solve linear programming approximations of the non-linear market-clearing problem (MC). More precisely, it proceeds as follows:

- Approximate $\mathcal{D}_j, j \in \mathcal{J}$ and $\mathcal{D}_s, s \in \mathcal{S}$ with $\tilde{\mathcal{D}}_j^{k_j}$ and $\tilde{\mathcal{D}}_s^{k_s}$, respectively.
- For $q_j \in \mathcal{D}_j, j \in \mathcal{J}$, replace $V_j(q_j) = V_j(\sum_{r=1}^{k_j} q_j^r\alpha_j^r)$ with $\sum_{r=1}^{k_j}\alpha_j^r V_j(q_j^r)$. Similarly, replace $C_s(q_s) = C_s(\sum_{r=1}^{k_s} q_s^r\beta_s^r)$ with $\sum_{r=1}^{k_s}\beta_s^r C_s(q_s^r)$.

These approximations yield the restricted master problem (MC-R):

$$\max \quad \sum_{j\in\mathcal{J}}\sum_{r=1}^{k_j} \alpha_j^r V_j(q_j^r) - \sum_{s\in\mathcal{S}}\sum_{r=1}^{k_s} \beta_s^r C_s(q_s^r) \tag{4}$$

$$\text{s.t.} \quad \sum_{j\in\mathcal{J}}\sum_{r=1}^{k_j} \alpha_j^r q_{j,l}^r - \sum_{s\in\mathcal{S}}\sum_{r=1}^{k_s} \beta_s^r q_{s,l}^r = 0, \quad l \in \mathcal{L} \tag{5}$$

$$\sum_{r=1}^{k_j} \alpha_j^r = 1, \quad j \in \mathcal{J} \tag{6}$$

$$\sum_{r=1}^{k_s} \beta_s^r = 1, \quad s \in \mathcal{S} \tag{7}$$

$$\alpha_j^r \geq 0, \quad r = 1, \dots k_j, \quad j \in \mathcal{J} \tag{8}$$

$$\beta_s^r \geq 0, \quad r = 1, \dots k_s, \quad s \in \mathcal{S} \tag{9}$$

Let $\{\tilde{\alpha}_j^r\}_{r=1,\dots,k_j, j\in\mathcal{J}}$, $\{\tilde{\beta}_s^r\}_{r=1,\dots,k_s, s\in\mathcal{S}}$ be an optimal basic solution of (MC-R) and $\{\tilde{\mu}_l\}_{l\in\mathcal{L}}$, $\{\tilde{\tau}_j\}_{j\in\mathcal{J}}$, and $\{\tilde{\tau}_s\}_{s\in\mathcal{S}}$ the corresponding dual multipliers. The optimality conditions for the restricted master problem are: $V_j(q_j^r) - \sum_{l\in\mathcal{L}} \tilde{\mu}_l q_{j,l}^r - \tilde{\tau}_j \leq 0, r = 1, \dots, k_j$, $j \in \mathcal{J}$ and $-C_s(q_s^r) + \sum_{l\in\mathcal{L}} \tilde{\mu}_l q_{s,l}^r - \tilde{\tau}_s \leq 0, r = 1, \dots, k_s$, $s \in \mathcal{S}$.

Thus, the generation of new feasible consumption levels $q_j^{k_j+1}$, $j \in \mathcal{J}$, or production levels $q_s^{k_s+1}$, $s \in \mathcal{S}$, that eventually improve the approximation can be done by pricing out sets $\mathcal{D}_j, j \in \mathcal{J}$, and $\mathcal{D}_s, s \in \mathcal{S}$; that is, by solving the sub-problems:

$$(\mathrm{SP})_j : \quad \max_{q_j \in \mathcal{D}_j} \{V_j(q_j) - \sum_{l\in\mathcal{L}} \tilde{\mu}_l q_{j,l} - \tilde{\tau}_j\}, j \in \mathcal{J};$$

and

$$(\mathrm{SP})_s : \quad \max_{q_s \in \mathcal{D}_s} \{-C_s(q_s) + \sum_{l\in\mathcal{L}} \tilde{\mu}_l q_{s,l} - \tilde{\tau}_s\}, s \in \mathcal{S}.$$

The DW decomposition principle has a classical economic interpretation as a decentralized planning procedure (Dantzig [13], chapter 23). A central authority (the headquarters) has to devise an optimal operation plan for an enterprise composed of several subsidiaries. Each subsidiary has private information concerning its technology and how it constrains its contribution to the overall plan. The headquarters deals with the constraints concerning the resource exchange between the subsidiaries. DW decomposition can be viewed as an iterative decision process in which the role of the central authority is to determine an optimal operation plan given a set of partial operation plans suggested by the subsidiaries and to announce corresponding dual prices, while the subsidiaries react to the announced prices by proposing new promising partial plans.

For the exchange economy considered in this paper, this principle translates into a two-phase auction mechanism:

- **Phase 1.**

  In this initialization phase, the market-maker asks the sellers and the buyers to submit the $|\mathcal{L}|+|\mathcal{J}|+|\mathcal{S}|$ bids required to build an initial restricted master problem.

  Denote by $k_j^{(0)}$ the number of bids submitted by buyer j during the initialization phase, and by $k_s^{(0)}$ the number of bids submitted by seller s.

- **Phase 2.**

  At iteration $n \geq 0$ of the auction:

  - Suppose $k_j^{(n)}$ and $k_s^{(n)}$ bids have been submitted, up to iteration n, by buyer j and seller s, respectively. The market-maker solves the restricted master problem (MC-R) and announces prices $\{\tilde{\mu}_l\}_{L\in\mathcal{L}}$, as well as multipliers $\{\tilde{\tau}_j\}_{j\in\mathcal{J}}$ and $\{\tilde{\tau}_s\}_{s\in\mathcal{S}}$ to the participants.
  - Each buyer $j, j \in \mathcal{J}$, determines a surplus-maximizing solution $q_j^{k_j^{(n)}}$ of $(\mathrm{SP})_j$. If $q_j^{k_j^{(n)}}$ improves on the approximation, buyer j submits bid $B_j^{(n)} = \{q_j^{k_j^{(n)}}; V(q_j^{k_j^{(n)}})\}$.
    Similarly, each seller $s, s \in \mathcal{S}$ determines a solution $q_s^{k_s^{(n)}}$ of $(\mathrm{SP})_s$. If $q_s^{k_s^{(n)}}$ improves the approximation, seller s submits bid $B_s^{(n)} = \{q_s^{k_s^{(n)}}; C(q_s^{k_s^{(n)}})\}$.

- **Stopping Criterion.**

  The auction may stop when:

  - The reduced-cost optimality condition is ϵ-satisfied for all bidder sub-problems, with ϵ being a small constant, which means no significant improvement by any additional bid is possible; or,
  - A maximum number of bidding rounds is attained.

  Otherwise, move to the next bidding round: $n \leftarrow n+1$.

5 APPLICATION TO A PROCUREMENT CASE

Our proposed auction mechanisms are illustrated on a more detailed model of multilateral multi-commodity markets presented in Bourbeau et al. [7]. This model has the

advantage of being closer to actual applications in procurement, especially in the context of regulated marketplaces for the trade of natural resources. We briefly present in the following the notation and the important elements of the model. We refer the reader interested in more details about the model to [7].

Participants in the market seek to trade a set of products. A product is a basic commodity with a specific physical denomination (e.g., a wood species). Products are generally not available in a "pure" state and come rather as part of lots that are "mixtures" of several products. Hence, let $\mathcal{K}$ be the set of basic products, $\mathcal{L}$ the set of lots, and b_l^k be the proportion of product k in lot l, $k \in \mathcal{K}$, $l \in \mathcal{L}$.

It is assumed for simplicity (but with no loss of generality) that each seller may only offer a single lot. Thus, a lot $l \in \mathcal{L}$ is attached to seller l and Q^l denotes the maximum quantity produced of that lot. The production cost function $C_l(.)$ of lot l is assumed to have a continuous, piecewise-linear, and strictly increasing marginal cost function $C_l'(.)$. On the buyer side, Bourbeau *et al.*'s model [7] accounts for the differences in quality among the various lots by considering: (i) a multiplicative adjustment coefficient r_j^l, which indicates that one unit of lot l is equivalent for buyer j to r_j^l units of a standard lot; and (ii) an additive coefficient s_j^l, which denotes how much more or less buyer j values, in absolute terms, a unit of lot l with respect to a unit of the standard lot. Furthermore, the model considers a unit transportation cost t_j^l between the seller producing lot l and buyer j. The latter's preference for a bundle $q_j = \{q_{j,l}\}_{l\in\mathcal{L}}$ can accordingly be expressed as $V_j(q_j) = U_j(\sum_{l\in\mathcal{L}} r_j^l q_{j,l}) + \sum_{l\in\mathcal{L}} (s_j^l - t_j^l)q_{j,l}$, where $U_j(.)$ is a utility function such that $U_j'(.)$ is continuous, piecewise-linear, and strictly decreasing. Buyers need also to express requirements regarding the composition of the lots they purchase. More specifically, let M_j^k and m_j^k denote respectively the maximum and minimum proportions of product k that buyer j may tolerate in the acquired lots, and Q^j the maximum total volume - expressed in terms of the standard lot - buyer j requires.

With the notation above, the market-clearing problem corresponds to the following formulation:

$$
\begin{aligned}
\max \quad & \sum_{j\in\mathcal{J}} U_j(\sum_{l\in\mathcal{L}} r_j^l q_{j,l}) + \sum_{l\in\mathcal{L}} (s_j^l - t_j^l) q_{j,l} - \sum_{l\in\mathcal{L}} C_l(q_l) \\
s.t. \quad & \sum_{j\in\mathcal{J}} q_{j,l} - q_l = 0, \quad l \in \mathcal{L} && (10)\\
& \sum_{l\in\mathcal{L}} r_j^l q_{j,l} \leq Q^j, \quad j \in \mathcal{J} && (11)\\
& m_j^k \sum_{l\in\mathcal{L}} r_j^l q_{j,l} \leq \sum_{l\in\mathcal{L}} b_l^k r_j^l q_{j,l}, \quad j \in \mathcal{J}, k \in \mathcal{K} && (12)\\
& \sum_{l\in\mathcal{L}} b_l^k r_j^l q_{j,l} \leq M_j^k \sum_{l\in\mathcal{L}} r_j^l q_{j,l}, \quad j \in \mathcal{J}, k \in \mathcal{K} && (13)\\
& 0 \leq q_l \leq Q^l, q_{j,l} \geq 0, \quad j \in \mathcal{J}, l \in \mathcal{L} && (14)
\end{aligned}
$$

Table 1: Characteristics of Problem Instances

Problem series	**Problem description**			
	# buyers	# lots	# products	$\Delta_m = \underline{M} - \overline{m}$ (%)
S01	50	100	3	30
S02	50	250	3	30
S03	100	50	3	30
S04	100	200	3	30
S05	100	500	3	30
S06	50	100	10	10
S07	50	250	10	10
S08	100	50	10	10
S09	100	200	10	10
S10	100	500	10	10

where $q_{j,l}$ denotes the quantity of lot l purchased by buyer j and q_l the total quantity of lot l procured by the corresponding seller.

The benchmark used is a mechanism based on the centralized market-clearing formulation (10-14), which assumes that the market-maker has access to complete information about the sellers and buyers valuation and cost functions, as well as their private technological constraints.

We have performed tests on several problem series made of instances obtained from a custom problem generator we have developed. Given values for the numbers of buyers, sellers (lots), and basic products, volumes $\{Q_j\}_{j\in\mathcal{J}}$ and $\{Q_l\}_{l\in\mathcal{L}}$, proportions $\{b_l^k\}_{l\in\mathcal{L},k\in\mathcal{K}}$, and tolerances $M_j^k, m_j^k, j \in \mathcal{J}, k \in \mathcal{K}$ are randomly generated according to continuous uniform distributions over pre-specified intervals. For the sake of simplicity, we considered *purely quadratic* buyer utility functions $U_j(.), j \in \mathcal{J}$ and seller cost functions $C_l(.), l \in \mathcal{L}$. This implies no loss of generality, since a simple transformation suggested in [7] allows to deal with a general piecewise-quadratic formulation as a purely quadratic one. Furthermore, our instances involved no transportation costs t_l^j or additive adjustment coefficients s_l^j, $j \in \mathcal{J}, l \in \mathcal{L}$. Table 1 displays the characteristics of the problem series considered in the study. The problem series vary according to (1) $|\mathcal{K}|$, the number of basic products; and (2) Δ_m, the minimum difference between tolerances M_j^k, m_j^k ($\Delta_m = \underline{M} - \overline{m}$, where $\underline{M}$ designates the minimum value M_j^k can take, and $\overline{m}$ the maximum value of m_j^k). These two parameters are important since they directly impact the number and forcefulness of constraints (13 and 12) in the market-clearing formulation. Each problem series consists of 10 randomly generated instances.

We have set up a DW auction based on our own basic implementation of DW column generation. The maximum number of rounds has been fixed to 400 rounds. The following metric is used: $GAP_{\text{dw}} = (Z_{\text{cent}} - Z_{\text{dw}})/Z_{\text{cent}}$, the gap between the optimal value

Z_{cent} obtained by the centralized allocation model (MC) and the lower bound obtained by Dantzig-Wolfe auction process Z_{dw}.

The experiments were carried out on a 64-processor, 64 Gigabytes of RAM Sun Enterprise 10000 operated under SunOS 5.8, with versions 8.0 and 1.2 of the CPLEX solver and the Concert library, respectively.

The behavior of the DW auction process is summarized in Table 2. The latter displays the number of bidding rounds considering two stopping criteria: (1) a less-than 10^{-5} gap GAP_{dw}, and (2) A less-then 10^{-6} reduced cost (RC) stopping criterion for the DW process, that is, no significant improvement by the new bids of the approximation of the objective during the pricing-out step. CPU times from the start of the auction process to its conclusion are also displayed in the table. The results indicate convergence on most instances within the 400 round limit.

In comparison with auctions based on Lagrangian relaxation we propose in Abrache et al. [1], the following preliminary observations can be made:

1. Auctions mechanisms based on bundle methods provide the only decent competition to DW, as simpler gradient-based auctions generally fail to converge within reasonable number of bidding round with an ϵ-feasible primal allocation.

2. Compared with bundle-based auctions, the following trade-off between the number of rounds and CPU times could be observed: much larger numbers of bidding rounds (between 1000 and 2000 for most problem instances) were needed for the bundle-based auction to converge to ϵ-feasible primal allocations, with an ϵ is the same order of magnitude than the tolerance used by the DW auction; on the other hand, the CPU times requirements scaled to the number of bidding rounds were sensibly lower than the DW ones (see Table 3 for a comparison of averages and standard deviation of CPU time to number of bidding round ratios). This could be explained by the computational bottleneck created by the solution of the master problem. Notice that the same observation has been made by other comparative studies on other types of problems (e.g., Vaidyanathan and Ahuja [26] for the multicommodity flow problem).

3. The DW-based auction has the significant advantage of being a primal method, working within the space of feasible allocations from the beginning of the auction until the end. This means the market-maker could stop the auction anytime with the guarantee that the allocation returned are always implementable, independent of the level of efficiency of the overall allocation.

4. The DW-based auction is relatively simpler to understand by both the market-maker and the bidder thanks to its intuitive economic interpretation. In contrast,

Table 2: Convergence of the DW Auction for Problem Series S01 to S10.

Problem	# Rounds		CPU	Problem	# Rounds		CPU
	$GAP_{\mathbf{dw}} \leq 10^{-5}$	RC Criterion	(sec.)		$GAP_{\mathbf{dw}} \leq 10^{-5}$	RC Criterion	(sec.)
S01Pb01	79	105	227.51	S06Pb01	95	119	328.87
S01Pb02	75	95	204.18	S06Pb02	104	132	377.2
S01Pb03	68	94	202.52	S06Pb03	99	124	348.2
S01Pb04	76	108	234.22	S06Pb04	107	145	401.39
S01Pb05	76	104	224.58	S06Pb05	105	135	387.13
S01Pb06	92	133	304.72	S06Pb06	157	178	556.59
S01Pb07	93	120	285.14	S06Pb07	152	180	573.19
S01Pb08	139	181	457.90	S06Pb08	104	135	412.07
S01Pb09	89	120	273.19	S06Pb09	102	132	391.97
S01Pb10	86	111	252.21	S06Pb10	92	114	342.38
S02Pb01	120	148	1488.44	S07Pb01	113	158	1851.33
S02Pb02	132	176	1859.91	S07Pb02	137	181	2107.41
S02Pb03	116	137	1391.85	S07Pb03	138	179	2131.75
S02Pb04	130	187	2029.14	S07Pb04	126	181	2250.27
S02Pb05	109	148	1516.85	S07Pb05	133	187	2209.31
S02Pb06	120	161	1670.56	S07Pb06	143	177	2117.90
S02Pb07	131	165	1802.64	S07Pb07	130	171	2055.09
S02Pb08	118	167	1740.40	S07Pb08	129	169	2008.47
S02Pb09	131	166	1864.28	S07Pb09	122	161	1858.03
S02Pb10	122	174	1896.38	S07Pb10	141	183	2230.64
S03Pb01	101	130	325.40	S08Pb01	133	166	496.68
S03Pb02	105	135	411.62	S08Pb02	116	142	473.58
S03Pb03	86	112	273.34	S08Pb03	126	161	530.64
S03Pb04	100	133	340.99	S08Pb04	128	151	462.29
S03Pb05	96	138	345.91	S08Pb05	117	147	443.78
S03Pb06	96	125	320.64	S08Pb06	137	178	513.41
S03Pb07	98	129	384.03	S08Pb07	118	145	413.02
S03Pb08	90	119	301.15	S08Pb08	133	174	588.35
S03Pb09	107	138	401.84	S08Pb09	136	162	530.70
S03Pb10	86	117	318.45	S08Pb10	116	147	425.78
S04Pb01	184	249	6737.06	S09Pb01	221	282	7817.13
S04Pb02	169	220	5546.10	S09Pb02	213	293	8163.64
S04Pb03	147	226	5513.94	S09Pb03	192	243	6355.45
S04Pb04	174	228	6016.38	S09Pb04	216	265	7735.9
S04Pb05	160	213	4995.58	S09Pb05	191	237	6589.59
S04Pb06	216	288	9131.96	S09Pb06	245	306	9535.88
S04Pb07	183	244	6733.79	S09Pb07	225	286	7236.46
S04Pb08	152	200	4941.07	S09Pb08	238	298	8553.55
S04Pb09	156	214	5706.89	S09Pb09	225	290	8578.7
S04Pb10	243	293	9873.75	S09Pb10	203	250	6726.14
S05Pb01	285	338	43334.1	S10Pb01	269	345	43360.27
S05Pb02	238	320	39982.90	S10Pb02	400	400	57209.84
S05Pb03	226	308	33954.93	S10Pb03	261	336	37137.53
S05Pb04	249	357	43696.49	S10Pb04	270	365	45529.23
S05Pb05	238	313	36781.61	S10Pb05	282	381	49178.12
S05Pb06	227	280	33363.27	S10Pb06	244	333	38499.70
S05Pb07	236	305	37194.54	S10Pb07	309	399	53485.34
S05Pb08	243	309	38088.24	S10Pb08	305	375	50591.40
S05Pb09	218	327	38237.93	S10Pb09	259	345	41305.67
S05Pb10	368	400	60557.18	S10Pb10	308	381	48641.40

bundle methods are mathematically much more sophisticated. Also, the implementation and tuning effort needed to successfully put them into practice is much higher.

6 CONCLUDING REMARKS

This paper has presented a new perspective of mathematical decomposition methods as iterative auctions in combinatorial exchanges of interdependent goods. We have focused on DW decomposition, and we have shown that it can be the basis of price-driven iterative mechanisms in which the participants progressively reveal their preferences to the market-maker. Numerical results obtained on a wood chip market case show relatively quick convergence to a near optimal allocation. These results are related to dual mechanisms

Table 3: Comparison of ratios CPU time to Nbr. rounds for DW and bundle-based auctions

Problem series	**Average and standard deviation of ratios CPU time to Nbr. rounds**	
	DW	Bundle
S01	(2.254, 0.117)	(0.600, 0.023)
S02	(10.574, 0.367)	(1.972, 0.061)
S03	(2.697, 0.204)	(0.622, 0.021)
S04	(27.088, 3.092)	(3.631, 0.131)
S05	(123.892, 10.417)	(13.845, 0.330)
S06	(2.940, 0.142)	(1.556, 0.145)
S07	(11.911, 0.249)	(5.716, 0.286)
S08	(3.099, 0.194)	(1.494, 0.133)
S09	(28.039, 1.626)	(11.304, 0.748)
S10	(126.502, 9.121)	(40.698, 1.014)

based on Lagrangian relaxation we developed in an earlier paper (Abrache et al. [1]) with an application to the same simulated market, in which we present sub-gradient and bundle-based iterative auctions. This comparison, albeit a preliminary one, allowed us to draw insightful conclusions on the degree of suitability of different mathematical programming decomposition methods to serve as the basis of practical iterative auction mechanisms. Yet, our conclusions cannot be generalized without a more comprehensive comparative study involving more problems of various structures. This study, along with the exploration of the more theoretically challenging issue of incentive compatibility, form the core of our future research on the topic.

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